

NO LOCAL L^1 SOLUTIONS FOR SEMILINEAR FRACTIONAL HEAT EQUATIONS

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ABSTRACT. We study the Cauchy problem for the semilinear fractional heat equation $u_t = \Delta^{\alpha/2}u + f(u)$ with non-negative initial value $u_0 \in L^q(\mathbb{R}^n)$ and locally Lipschitz, non-negative source term f . For f satisfying the Osgood-type condition $\int_1^\infty \frac{ds}{f(s)} = \infty$, we show that there exist initial conditions such that the equation has no local solution in $L^1_{loc}(\mathbb{R}^n)$.

1. Introduction

In this paper, we consider the Cauchy problem for the semilinear fractional heat equation

$$u_t = \Delta^{\alpha/2}u + f(u), \quad u(0) = u_0 \in L^q(\mathbb{R}^n), \quad q \geq 1, \quad (1.1)$$

where $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ denotes the fractional Laplacian defined by the Fourier transform

$$(\mathcal{F}(-\Delta)^{\alpha/2}u)(\xi) = |\xi|^\alpha \mathcal{F}(u)(\xi),$$

where $0 < \alpha \leq 2$, \mathcal{F} denotes the Fourier transform.

We assume the following conditions hold for the initial value u_0 and source term f :

- (A1) $u_0 \geq 0$ and $u_0 \in L^q(\mathbb{R}^n)$;
- (A2) $f : [0, \infty) \rightarrow [0, \infty)$ is locally Lipschitz continuous, non-decreasing, $f(0) = 0$, $f > 0$ on $(0, \infty)$;
- (A3) f satisfies the Osgood-type condition

$$\int_1^\infty \frac{ds}{f(s)} = \infty. \quad (1.2)$$

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Recall ordinary differential equations of the form

$$u_t = f(u), \quad (1.3)$$

where f is positive, continuous and satisfies (1.2). It is well known that this Osgood-type condition is necessary and sufficient for the global existence of solutions of (1.3) with initial data $u_0 \geq 0$.

For $\alpha = 2$, (1.1) reduces to

$$u_t = \Delta u + f(u), \quad u(0) = u_0 \in \mathbb{R}^n. \quad (1.4)$$

When f is locally Lipschitz continuous, by Theorem 1.4 in [5], there exists a $t_{max} \leq \infty$ such that a unique mild solution of (1.4) exists on $[0, t_{max})$, if $t_{max} < \infty$, then $\lim_{t \uparrow t_{max}} \|u(t)\| = \infty$. For non-negative $u_0 \in L^\infty(\mathbb{R}^n)$ and f satisfying the Osgood-type condition (1.2), we can get the global existence of (1.4) (See, for example, Remark 5.1 in [10]). However, when the initial data u_0 is singular or unbounded, the question whether (1.4) has global existence hasn't been settled until the appearance of [1]. Existence results for (1.4) is considered in [1] under the conditions that the source term f satisfies conditions (A2) and (A3). The authors showed that there are initial conditions satisfying (A1) for which there is no local integral solution (See the following Definition 1) of (1.4) that remains in $L^1_{loc}(\mathbb{R}^n)$.

Definition 1. (See [9], p.78) Given f non-negative and $u_0 \geq 0$, we say that u is a local integral solution of (1.4) on $[0, T)$ if $u : \mathbb{R}^n \times [0, T) \rightarrow [0, \infty]$ is measurable, finite almost everywhere, and

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(u(s))ds$$

holds almost everywhere on $\mathbb{R}^n \times [0, T)$, where

$$(S(t)u_0)(x) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} u_0(y) dy$$

is a classical solution of the linear heat equation

$$v_t = \Delta v, \quad v(0) = u_0.$$

From Definition 1, we see that if u is a classical solution (or mild solution), then it is an integral solution. Non-existence integral solutions implies the non-existence of classical solutions (or mild solutions). The non-existence of a local solution is ‘instantaneous blow-up’ in some sense (see, e.g., [11]).

In fact, the Osgood-type condition (1.2) is not necessary for global existence of solutions for (1.4). Fujita [6, 7] studied the initial value problem

$$u_t = \Delta u + u^p, \quad u(0) = u_0 \in \mathbb{R}^n, \quad (1.5)$$

where $u_0 \geq 0$, $p > 1$. He proved that if $p > 2/n + 1$, then global solutions exists for small initial data u_0 . Weissler [8] considered the problem (1.5), in the case $p > 2/n + 1$, $\|u_0\|_{L^{n(p-1)/2}}$ is sufficiently small, the global solution is obtained. It is clear that the source term $f(u) = u^p$ ($p > 1$) doesn’t satisfy the Osgood-type condition (1.2).

In recent years, a great deal of attention has been paid to fractional differential equations with fractional Laplacian due to their many applications in mathematics, physics, biology, see for instance [13, 14, 15, 17, 16] and references therein. It is nature to consider the question whether the Osgood-type condition (1.2) guarantees the global existence of solutions for semilinear fractional heat equations (1.1) with singular (or unbounded) initial data. We answer this question in the negative.

The paper is organized as follows. In Section 2, we present some estimates for the linear fractional heat equation. In Section 3, we construct a function which satisfies the Osgood-type condition. In Section 4, we prove the non-existence results for (1.1) with unbounded initial data.

2. Estimates for some solutions of the fractional heat equation

For any $r > 0$, let $B_r(x)$ be the Euclidean ball in \mathbb{R}^n of radius R centred at x , S^{n-1} the unit sphere and ω_n the volume of the unit ball $B_1(0)$. Let $p(t, x, y)$ be the heat kernel of $\Delta^{\alpha/2}$ on \mathbb{R}^n . Throughout this paper, we use c_0, c_1, c_2, \dots to

denote generic constants, which may change from line to line. For two nonnegative functions f_1 and f_2 , the notion $f_1 \asymp f_2$ means that $c_1 f_2(x) \leq f_1(x) \leq c_2 f_2(x)$, where c_1, c_2 are positive constants. It is well known that (see, e.g., [3, 4, 2])

$$p(t, x, y) \asymp \left(t^{-n/\alpha} \wedge \frac{t}{|x - y|^{n+\alpha}} \right),$$

that is, there exist constants c_1, c_2 such that for $t > 0, x, y \in \mathbb{R}^n$,

$$c_1 \left(t^{-n/\alpha} \wedge \frac{t}{|x - y|^{n+\alpha}} \right) \leq p(t, x, y) \leq c_2 \left(t^{-n/\alpha} \wedge \frac{t}{|x - y|^{n+\alpha}} \right), \quad (2.1)$$

where c_1 and c_2 are positive constants depending on α .

From (2.1), it follows

$$\frac{c_3 t}{(t^{1/\alpha} + |y - x|)^{n+\alpha}} \leq p(t, x, y) \leq \frac{c_4 t}{(t^{1/\alpha} + |y - x|)^{n+\alpha}}, \quad (2.2)$$

where $t > 0, x, y \in \mathbb{R}^n$, c_3 and c_4 are positive constants depending on α .

Definition 2. *u is said to be a local integral solution of (1.1) on $[0, T)$, if $u : \mathbb{R}^n \times [0, T) \rightarrow [0, \infty)$ is measurable, finite almost everywhere and*

$$u(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(t-s)f(u(s))ds \quad (2.3)$$

holds almost everywhere in $\mathbb{R}^n \times [0, T)$, where $(S_\alpha(t)u_0)(x)$ is a classical solution of the linear fractional heat equation

$$w_t = \Delta^{\alpha/2} w, \quad w(0) = u_0, \quad (2.4)$$

and

$$(S_\alpha(t)u_0)(x) = \int_{\mathbb{R}^n} p(t, x, y)u_0(y)dy, \quad (2.5)$$

where $p(t, x, y)$ is the heat kernel of $\Delta^{\alpha/2}$ on \mathbb{R}^n .

Proposition 1. *Let $\beta \in (0, n)$ and $R > 1$. Assume that $u_0 \in L^1(\mathbb{R}^n)$ be the non-negative, radially symmetric function given by*

$$u_0(x) = |x|^{-\beta} \chi_R := \begin{cases} |x|^{-\beta}, & |x| \leq R, \\ 0, & |x| > R. \end{cases}$$

Let $w(t) = S_\alpha(t)u_0$ and

$$M = \min\{w(\tilde{x}, t) : \tilde{x} \in S^{n-1}, 0 \leq t \leq 1\}. \quad (2.6)$$

If $\gamma \in (0, 1/\alpha)$, then for any $\phi \geq c_3M/c_4$,

$$w(x, t) \geq \phi \text{ for } |x| \leq t^\gamma \text{ and } 0 < t \leq (c_4\phi/c_3M)^{-1/\beta\gamma}, \quad (2.7)$$

where c_3, c_4 are the same constants as in (2.2).

Proof. For any $0 < t \leq 1$, $x = t^\gamma \tilde{x} \in \partial B_n(t^\gamma)$, $\tilde{x} \in S^{n-1}$,

$$\begin{aligned} w(x, t) &= w(t^\gamma \tilde{x}, t) \\ &\geq \frac{c_3}{t^{n/\alpha}} \int_{\mathbb{R}^n} \frac{u_0(y)}{(1 + t^{-1/\alpha}|y - x|)^{n+\alpha}} dy \\ &= \frac{c_3}{t^{n/\alpha}} \int_{\mathbb{R}^n} \frac{u_0(y)}{(1 + t^{-1/\alpha}|y - t^\gamma \tilde{x}|)^{n+\alpha}} dy \\ &= \frac{c_3}{t^{n/\alpha}} \int_{\mathbb{R}^n} \frac{u_0(t^\gamma z) t^{n\gamma}}{(1 + t^{\gamma-1/\alpha}|z - \tilde{x}|)^{n+\alpha}} dz \\ &= \frac{c_3}{t^{n/\alpha}} \int_{B_{t^{-\gamma}R}(0)} \frac{|t^\gamma z|^{-\beta} t^{n\gamma}}{(1 + t^{\gamma-1/\alpha}|z - \tilde{x}|)^{n+\alpha}} dz \\ &= \frac{t^{-\beta\gamma} c_3}{t^{n/\alpha}} \int_{B_{t^{-\gamma}R}(0)} \frac{|z|^{-\beta} t^{n\gamma}}{(1 + t^{\gamma-1/\alpha}|z - \tilde{x}|)^{n+\alpha}} dz \\ &= \frac{t^{-\beta\gamma} c_3}{t^{n(\frac{1}{\alpha}-\gamma)}} \int_{B_{t^{-\gamma}R}(0)} \frac{|z|^{-\beta}}{(1 + t^{\gamma-1/\alpha}|z - \tilde{x}|)^{n+\alpha}} dz \\ &\geq \frac{t^{-\beta\gamma} c_3}{t^{n(\frac{1}{\alpha}-\gamma)}} \int_{B_R(0)} \frac{|z|^{-\beta}}{(1 + t^{\gamma-1/\alpha}|z - \tilde{x}|)^{n+\alpha}} dz \\ &= \frac{t^{-\beta\gamma} c_3}{t^{n(\frac{1}{\alpha}-\gamma)}} \int_{\mathbb{R}^n} \frac{u_0(z)}{(1 + t^{\gamma-1/\alpha}|z - \tilde{x}|)^{n+\alpha}} dz. \end{aligned} \quad (2.8)$$

Since $\gamma \in (0, 1/\alpha)$, we have $0 < \gamma\alpha < 1$. Then from (2.8), it follows that

$$w(x, t) \geq \frac{t^{-\beta\gamma} c_3}{t^{n(\frac{1}{\alpha}-\gamma)}} \int_{\mathbb{R}^n} \frac{u_0(z)}{(1 + t^{\gamma-1/\alpha}|z - \tilde{x}|)^{n+\frac{1}{\alpha}-\gamma}} dz. \quad (2.9)$$

This together with (2.5) and (2.2) yield

$$w(x, t) \geq \frac{c_3}{c_4} t^{-\beta\gamma} w(\tilde{x}, t^{\alpha/(1-\alpha\gamma)}). \quad (2.10)$$

For $0 < t \leq 1$ and $|\tilde{x}| = 1$, by (2.6),

$$w(\tilde{x}, t^{\alpha/(1-\alpha\gamma)}) \geq M.$$

Note that w is radially symmetric and decreasing in the radial variable, then

$$w(x, t) \geq \frac{c_3}{c_4} M t^{-\beta\gamma} \text{ for all } |x| \leq t^\gamma.$$

Therefore, we obtain the conclusion. \square

3. A family of functions f satisfying the Osgood-type condition

In this section, we will construct a family of functions depending on a parameter $k > 1$, which satisfy (A2) and (A3).

For $\alpha \in (1, 2]$ and $k > 1$, choose $\phi_0 > \alpha^{1/(k-1)}$ and define the sequence ϕ_i by $\phi_{i+1} = \phi_i^k$.

Define $f : [0, \infty) \rightarrow [0, \infty)$ by

$$f(s) = \begin{cases} (1 - \phi_0^{1-k})s^k, & s \in J_0 := [0, \phi_0], \\ \phi_i - \phi_{i-1}, & s \in I_i := (\phi_{i-1}, \phi_i/\alpha], \quad i \geq 1, \\ l_i(s), & s \in J_i := (\phi_i/\alpha, \phi_i], \quad i \geq 1, \end{cases} \quad (3.1)$$

where l_i denotes the linear interpolated function between the values of f at ϕ_i/α and ϕ_i . It is clear that f satisfies (A2). For every $i \geq 1$, it is easy to see

$$1 < \phi_{i-1} < \phi_i/\alpha, \quad i \geq 1, \quad (3.2)$$

and

$$\lim_{i \rightarrow \infty} \phi_i = \infty. \quad (3.3)$$

We have

$$\begin{aligned} \int_1^\infty \frac{ds}{f(s)} &\geq \sum_i \int_{I_i} \frac{ds}{f(s)} \\ &= \sum_{i=1}^\infty \frac{\phi_i/\alpha - \phi_{i-1}}{\phi_i - \phi_{i-1}} \\ &= \frac{1}{\alpha} \sum_{i=1}^\infty \left(1 - \frac{(\alpha - 1)\phi_{i-1}}{\phi_i - \phi_{i-1}}\right) \\ &= \frac{1}{\alpha} \sum_{i=1}^\infty \left(1 - \frac{(\alpha - 1)}{\phi_{i-1}^{k-1} - 1}\right). \end{aligned} \quad (3.4)$$

Since $\lim_{i \rightarrow \infty} \left(1 - \frac{(\alpha-1)}{\phi_{i-1}^{k-1}-1}\right) = 1$, by (3.4), we obtain

$$\int_1^\infty \frac{ds}{f(s)} = \infty.$$

Note that $f(s)$ is bounded above by $\alpha^k s^k$ for $s \geq 0$. In fact, for $s \in [0, \phi_0]$, it is obvious; for $s \in I_i = (\phi_{i-1}, \phi_i/\alpha]$, $f(s) = \phi_i - \phi_{i-1} \leq \phi_{i-1}^k \leq s^k$; for $s \in J_i = (\phi_i/\alpha, \phi_i)$, $f(s) \leq \phi_{i+1} - \phi_i \leq \phi_i^k \leq \alpha^k s^k$.

Define the function $\tilde{f} : [0, \infty) \rightarrow [0, \infty)$ as

$$\tilde{f}(s) = \begin{cases} 0, & s \in J_0 := [0, \phi_0], \\ \phi_i - \phi_{i-1}, & s \in I_i \cup J_i := (\phi_{i-1}, \phi_i], \quad i \geq 1, \end{cases} \quad (3.5)$$

we see that $\tilde{f} = f$ on I_i , $f \geq \tilde{f}$ on J_i . Thus $f \geq \tilde{f}$ on $[0, \infty)$.

4. Non-existence of local solutions

Lemma 1. *If $\alpha \in (1, 2]$ and $k > 1 + \alpha/n$, then there exists a non-negative $u_0 \in L^1(\mathbb{R}^n)$ such that (1.1) has no local integral solution which is bounded in $L^1(\mathbb{R}^n)$.*

Proof. Since $k > 1 + \alpha/n$, we can choose $\beta \in (0, n)$ such that $k > (n + \alpha)/\beta$. Let u be a local integral solution of (1.1) with f constructed as (3.1) and $u_0 = |x|^{-\beta} \chi_R$. Since $f \geq 0$, $u_0 \geq 0$, f is non-decreasing, from (2.3), it follows that $u(t) \geq S_\alpha(t)u_0$ for all $t \geq 0$. Therefore,

$$u(t) \geq S_\alpha(t)u_0 + \int_0^t S_\alpha(t-s)f(S_\alpha(s)u_0)ds. \quad (4.1)$$

Since $\int_{\mathbb{R}^n} p(t, x, y)dx = 1$, we see that $S_\alpha(t)$ is L^1 norm-preserving. By (4.1) and Fubini's Theorem,

$$\begin{aligned} \|u(t)\|_{L^1} &= \int_{\mathbb{R}^n} u(t)dx \\ &\geq \int_{\mathbb{R}^n} S_\alpha(t)u_0dx + \int_{\mathbb{R}^n} \int_0^t S_\alpha(t-s)f(S_\alpha(s)u_0)dsdx \\ &= \int_{\mathbb{R}^n} u_0dx + \int_0^t \int_{\mathbb{R}^n} f(S_\alpha(s)u_0)dsdx \\ &\geq \int_0^t \int_{\mathbb{R}^n} f(S_\alpha(s)u_0)dxds. \end{aligned} \quad (4.2)$$

Since $k > (n + \alpha)/\beta$, we can choose $\gamma \in (0, 1/\alpha)$ such that $k > (n\gamma + 1)/\beta\gamma$. Set $w(x, t) = (S(t)u_0)(x)$. For sufficient large i , by (3.3), we have $\phi_i \geq M$ and $(c_4\phi_i/c_3M)^{-1/\beta\gamma} \leq t$, where M is as in (2.6) and c_3, c_4 are the same constants as in (2.2). Then by Proposition 1, for $|x| \leq s^\gamma$, $0 < s \leq (c_4\phi_i/c_3M)^{-1/\beta\gamma}$, we have $w(x, s) \geq \phi_i$. Since $f \geq \tilde{f}$ on $[0, \infty)$, $\phi_{i+1} = \phi_i^k$, we obtain

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^n} f(w(x, s)) dx ds &\geq \int_0^{(c_4\phi_i/c_3M)^{-1/\beta\gamma}} \int_{\mathbb{R}^n} f(w(x, s)) dx ds \\
&\geq \int_0^{(c_4\phi_i/c_3M)^{-1/\beta\gamma}} \int_{\{x: |x| \leq s^\gamma\}} \tilde{f}(w(x, s)) dx ds \\
&= \int_0^{(c_4\phi_i/c_3M)^{-1/\beta\gamma}} \int_{\{x: |x| \leq s^\gamma\}} (\phi_{i+1} - \phi_i) dx ds \\
&\geq \frac{(\alpha - 1)\phi_i^k}{\alpha} \int_0^{(c_4\phi_i/c_3M)^{-1/\beta\gamma}} \int_{\{x: |x| \leq s^\gamma\}} dx ds \\
&= \frac{(\alpha - 1)\phi_i^k}{\alpha} \int_0^{(c_4\phi_i/c_3M)^{-1/\beta\gamma}} \omega_n s^{\gamma n} ds \\
&= \frac{(\alpha - 1)\omega_n}{\alpha(\gamma n + 1)} \left(\frac{c_4}{c_3M}\right)^{-(rn+1)/\beta\gamma} \phi_i^{k-(rn+1)/\beta\gamma}. \tag{4.3}
\end{aligned}$$

By (3.3) and note that $k > (n\gamma + 1)/\beta\gamma$, we have $\phi_i^{k-(rn+1)/\beta\gamma} \rightarrow \infty$ as $i \rightarrow \infty$. Therefore, for such u_0 , there is no integral solution of (1.1) which is bounded in $L^1(\mathbb{R}^n)$. \square

Theorem 1. *Let $\alpha \in (1, 2]$ and $q \in [1, \infty)$. If $k > q(1 + \alpha/n)$, then there exists a non-negative $u_0 \in L^q(\mathbb{R}^n)$ such that (1.1) has no local integral solution which is in $L^1_{loc}(\mathbb{R}^n)$.*

Proof. Since $k > q(1 + \alpha/n)$, we can choose $\beta \in (0, n/q)$ such that $k > (n + \alpha)/\beta$. Then choose $\gamma \in (0, 1/\alpha)$ such that $k > (n\gamma + 1)/\beta\gamma$. Let u be a local integral solution of (1.1) with f constructed as (3.1). Fix $t_0 \in (0, 1)$, for $t \in (0, t_0)$, choose i sufficiently large such that $\tilde{t} := (c_4\phi_i/c_3M)^{-1/\beta\gamma} \leq t$, where M is as in (2.6) and c_3, c_4 are the same constants as in (2.2). Set $w(x, t) = (S(t)u_0)(x)$ and let $B_\rho(0)$ be

the ball of radius $\rho > 1$ centred at 0. By (4.1), we have

$$\begin{aligned}
\int_{B_\rho(0)} u(t) dx &\geq \int_{B_\rho(0)} \int_0^{\tilde{t}} [S_\alpha(t-s)f(w(\cdot, s))](x) ds dx \\
&= \int_0^{\tilde{t}} \int_{B_\rho(0)} \int_{\mathbb{R}^n} p(t-s, x, y) f(w(y, s)) dy dx ds \\
&= \int_0^{\tilde{t}} \int_{\mathbb{R}^n} \int_{B_\rho(0)} p(t-s, x, y) f(w(y, s)) dx dy ds \\
&\geq \int_0^{\tilde{t}} \int_{|y| \leq s^\gamma} \int_{B_\rho(0)} p(t-s, x, y) \frac{(\alpha-1)\phi_{i+1}}{\alpha} dx dy ds \\
&= \frac{(\alpha-1)\phi_{i+1}}{\alpha} \int_0^{\tilde{t}} \int_{|y| \leq s^\gamma} \int_{B_\rho(0)} p(t-s, x, y) dx dy ds. \tag{4.4}
\end{aligned}$$

By (2.2), we have

$$\int_{B_\rho(0)} p(t-s, x, y) dx \geq c \int_{B_\rho(0)} \frac{t-s}{((t-s)^{1/\alpha} + |y-x|)^{n+\alpha}} dx. \tag{4.5}$$

Since $0 < s \leq \tilde{t} < 1$, then $|y| \leq s^\gamma < 1$. The right hand integral is radially and decreasing with $|y|$. For $|x| \leq \rho$ ($\rho > 1$) and $(t-s)^{-1/\alpha} > 1$, choosing any unit vector τ , by (4.5),

$$\begin{aligned}
\int_{B_\rho(0)} p(t-s, x, y) dx &\geq c \int_{B_\rho(\tau)} \frac{t-s}{((t-s)^{1/\alpha} + |z|)^{n+\alpha}} dz \\
&= c \int_{B_\rho((t-s)^{-1/\alpha}\tau)} \frac{1}{(1+|v|)^{n+\alpha}} dv \\
&\geq c \int_{B_\rho(\tau)} \frac{1}{(1+|v|)^{n+\alpha}} dv \\
&\geq \tilde{c}, \tag{4.6}
\end{aligned}$$

where \tilde{c} is a positive constant.

By (4.4), (4.6), we obtain

$$\begin{aligned}
\|u(t)\|_{L^1(B_\rho(0))} &\geq \frac{\tilde{c}(\alpha-1)\phi_{i+1}}{\alpha} \int_0^{t_0} \int_{|y| \leq s^\gamma} dy ds \\
&\geq \frac{\tilde{c}(\alpha-1)\omega_n\phi_{i+1}}{\alpha} \int_0^{\tilde{t}} s^{n\gamma} ds \\
&\geq \bar{c}\phi_{i+1}\tilde{t}^{n\gamma+1} \\
&\geq \bar{c}\phi_i^{k-(n\gamma+1)/\beta\gamma} \rightarrow \infty \tag{4.7}
\end{aligned}$$

as $i \rightarrow \infty$. The proof is complete. \square

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